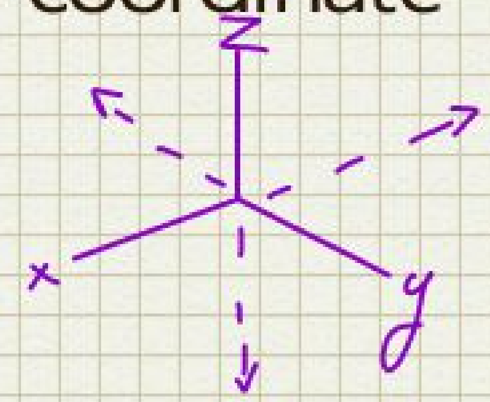
Multivariable Calculus

We begin to look at a three dimensional coordinate system with axes x y and z.
Review : Derivatives and Integrals... FTC
Derivatives totes talk about rate of change at a point instaneously. An integral describes area between a given function on an interval.

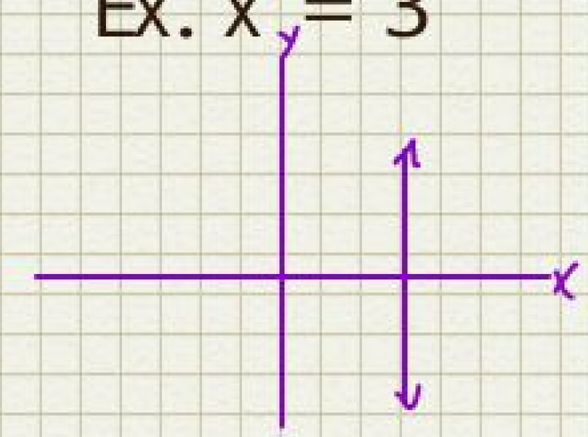
In 3d space, we use the coordinate system x,y,z with cooridinates (a,b,c) to uniquely represent a point in space. We divide this plane into Octants. There are 8 octants in this coordinate system.

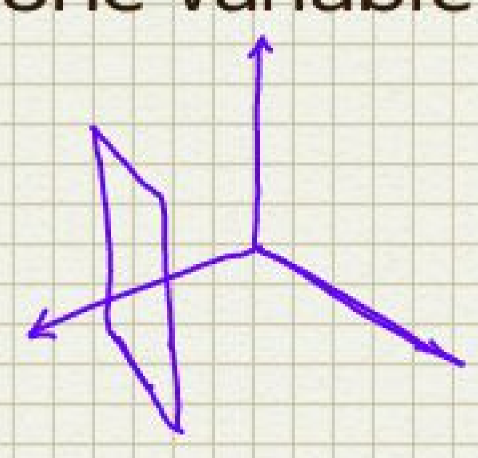


The horizontal should be X.
The other horizontal should be Y.
Right hand rule, the verical axis should be Z.

Talking about projections of planes from the 3d plane.

In R2, a value for one variable is a line, but in R3 a value for one variable represents a plane.
Ex. x = 3



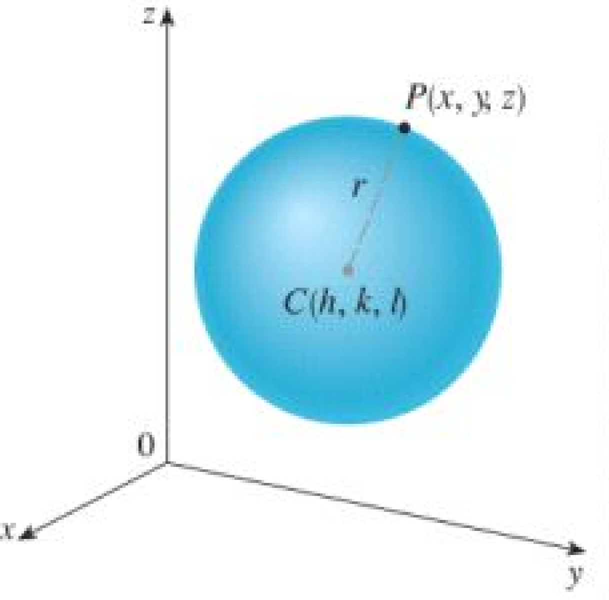


Equation of a Sphere An equation of a sphere with center $C(h, k, l)$ and radius r is

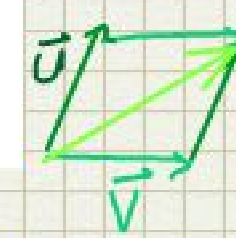
$$(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2$$

In particular, if the center is the origin O, then an equation of the sphere is

$$x^2 + y^2 + z^2 = r^2$$



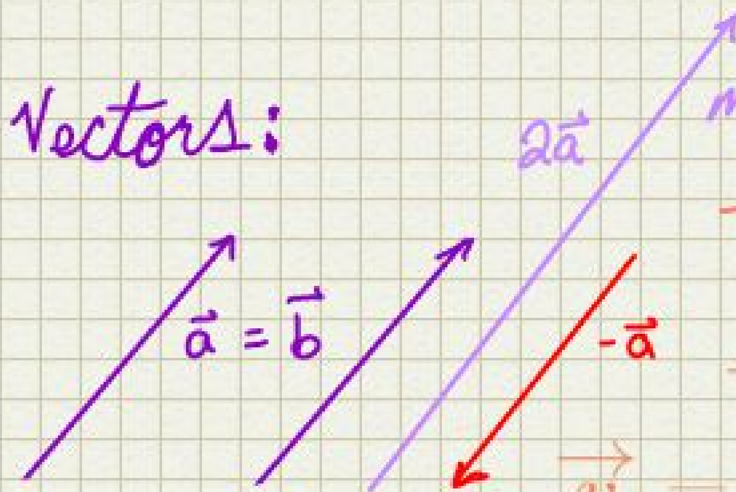
$\vec{u} + \vec{v}$



Vectors:

scalar multiplication

tip to tail



$$\overrightarrow{u} = \langle a, b \rangle$$

$$\overrightarrow{v} = \langle c, d \rangle$$

$$\overrightarrow{u} + \overrightarrow{v} = \langle a+c, b+d \rangle$$

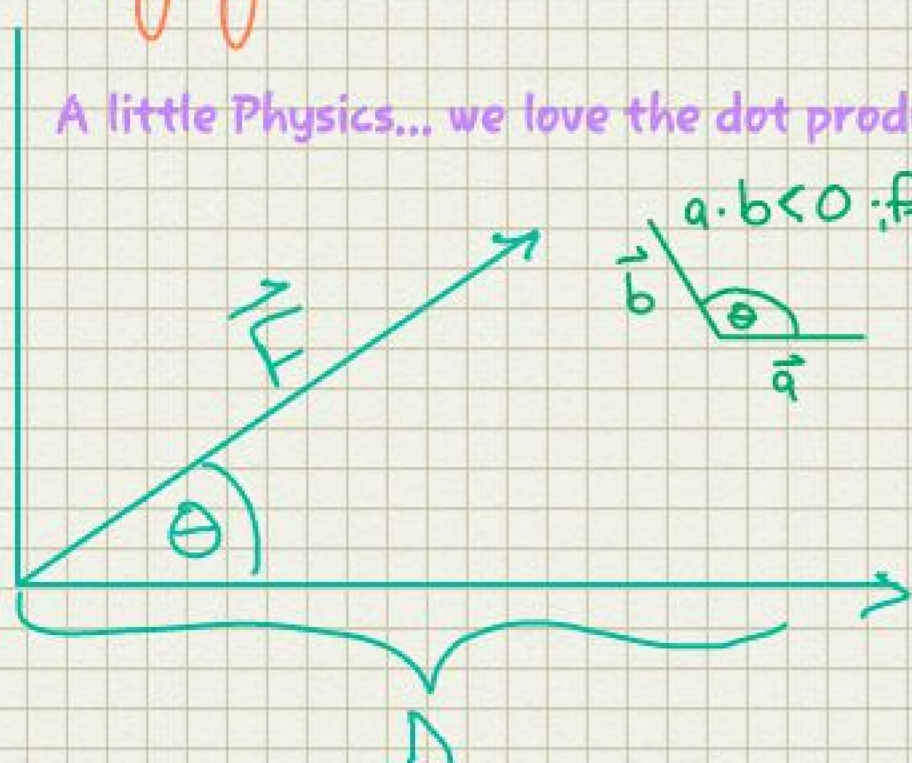
Equal vectors have equal magnitude and direction... Think slopes parallel

$\underbrace{\langle a, b \rangle}_{2d}$ $\underbrace{\langle a, b, c \rangle}_{3d}$

Vectors are described by their components

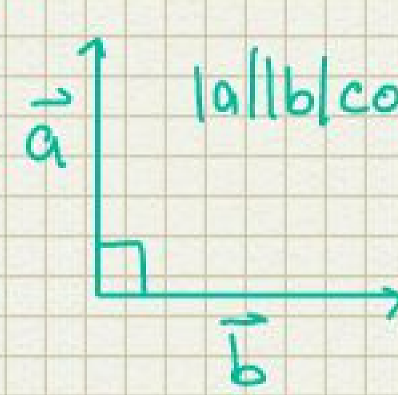
Multiplying Vectors {4d multiplying 8d}

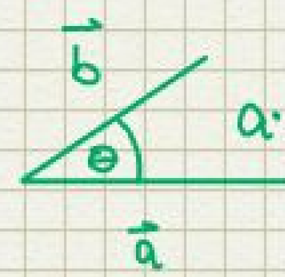
A little Physics... we love the dot product!



$a \cdot b < 0$:ff

$|a||b|\cos 90^\circ = 0$





$a \cdot b > 0$

$\text{Work} = |F||D|\cos\theta$

The dot Product

$|a||b|\cos\theta = a\cdot b$

$(a_x b_x) + (a_y b_y) + (a_z b_z) = a\cdot b$

$\cos\theta = \frac{A\cdot b}{|a||b|}$ $\theta = \cos^{-1}\left(\frac{A\cdot b}{|A||b|}\right)$

Scalar projection of **b** onto **a**: $\text{comp}_{\mathbf{a}}\,\mathbf{b} = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}$

Vector projection of **b** onto **a**: $\text{proj}_{\mathbf{a}}\,\mathbf{b} = \left(\dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}\right)\dfrac{\mathbf{a}}{|\mathbf{a}|} = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\,\mathbf{a}$

Properties of the Dot Product If **a**, **b**, and **c** are vectors in V_3 and c is a scalar, then

1. $\mathbf{a}\cdot\mathbf{a} = |\mathbf{a}|^2$
2. $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}$
3. $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{c}$
4. $(c\mathbf{a})\cdot\mathbf{b} = c(\mathbf{a}\cdot\mathbf{b}) = \mathbf{a}\cdot(c\mathbf{b})$ SCAle out
5. $\mathbf{0}\cdot\mathbf{a} = 0$

Le Cross Product

Definition If **a** and **b** are nonzero three-dimensional vectors, the **cross product** of **a** and **b** is the vector

$$\mathbf{a}\times\mathbf{b} = (|\mathbf{a}||\mathbf{b}|\sin\theta)\,\mathbf{n}$$

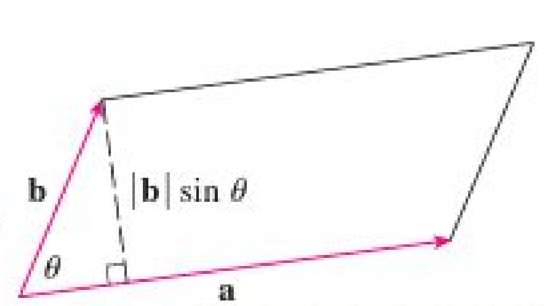
where θ is the angle between **a** and **b**, $0 \leqslant \theta \leqslant \pi$, and **n** is a unit vector perpendicular to both **a** and **b** and whose direction is given by the **right-hand rule**: If the fingers of your right hand curl through the angle θ from **a** to **b**, then your thumb points in the direction of **n**. (See Figure 3.)

The vector product is NOT commutative. i x j = k but j x i = k

Properties of the Cross Product If **a**, **b**, and **c** are vectors and c is a scalar, then

1. $\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a}$
2. $(c\mathbf{a})\times\mathbf{b} = c(\mathbf{a}\times\mathbf{b}) = \mathbf{a}\times(c\mathbf{b})$ SCAle out
3. $\mathbf{a}\times(\mathbf{b}+\mathbf{c}) = \mathbf{a}\times\mathbf{b} + \mathbf{a}\times\mathbf{c}$
4. $(\mathbf{a}+\mathbf{b})\times\mathbf{c} = \mathbf{a}\times\mathbf{c} + \mathbf{b}\times\mathbf{c}$

The length of the cross product $\mathbf{a}\times\mathbf{b}$ is equal to the area of the parallelogram determined by **a** and **b**.



Crossing 3d Vectors (in $\mathbb{R}^3$)

$$i \to j \to k \to i$$

2 If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then

$$\mathbf{a} \times \mathbf{b} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle$$

9.4) 3, 4, 7, 8, 10, 11, 13, 15, 19, 23, 24, 27, 28, 41

Some linear Algebra

2×2 determinant

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

"3×3 determinant"

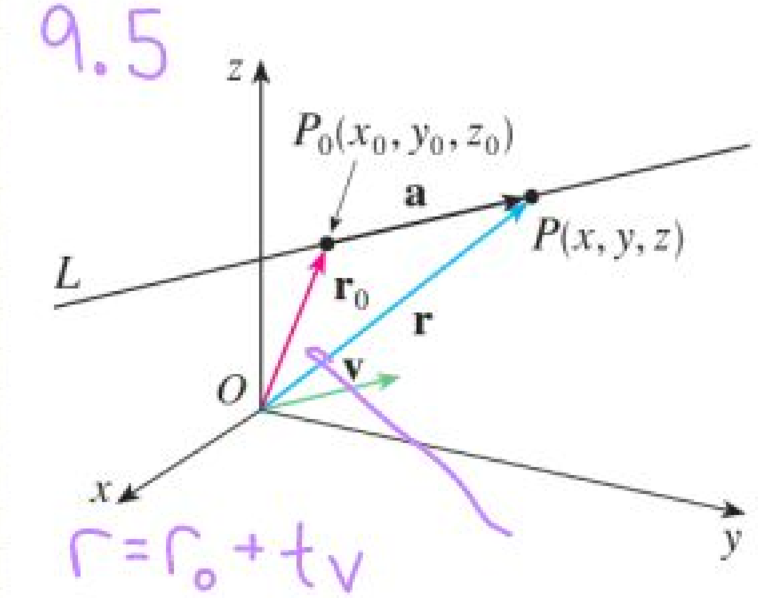
$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \quad \boxed{a_1 \begin{bmatrix} b_2 & b_3 \\ c_2 & c_3 \end{bmatrix} - a_2 \begin{bmatrix} b_1 & b_3 \\ c_1 & c_3 \end{bmatrix} + a_3 \begin{bmatrix} b_1 & b_2 \\ c_1 & c_2 \end{bmatrix}}$$

$$a = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$

$$b = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$$

$$\begin{bmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} = (a_2b_3 - a_3b_2)\hat{i} - (a_1b_3 - a_3b_1)\hat{j} + (a_1b_2 - a_2b_1)\hat{k}$$

9.5



$r = r_0 + tv$

FIGURE 1

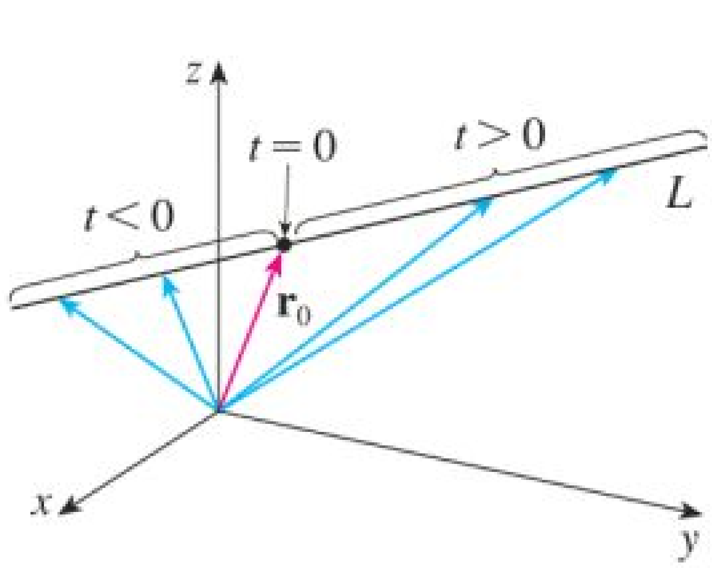
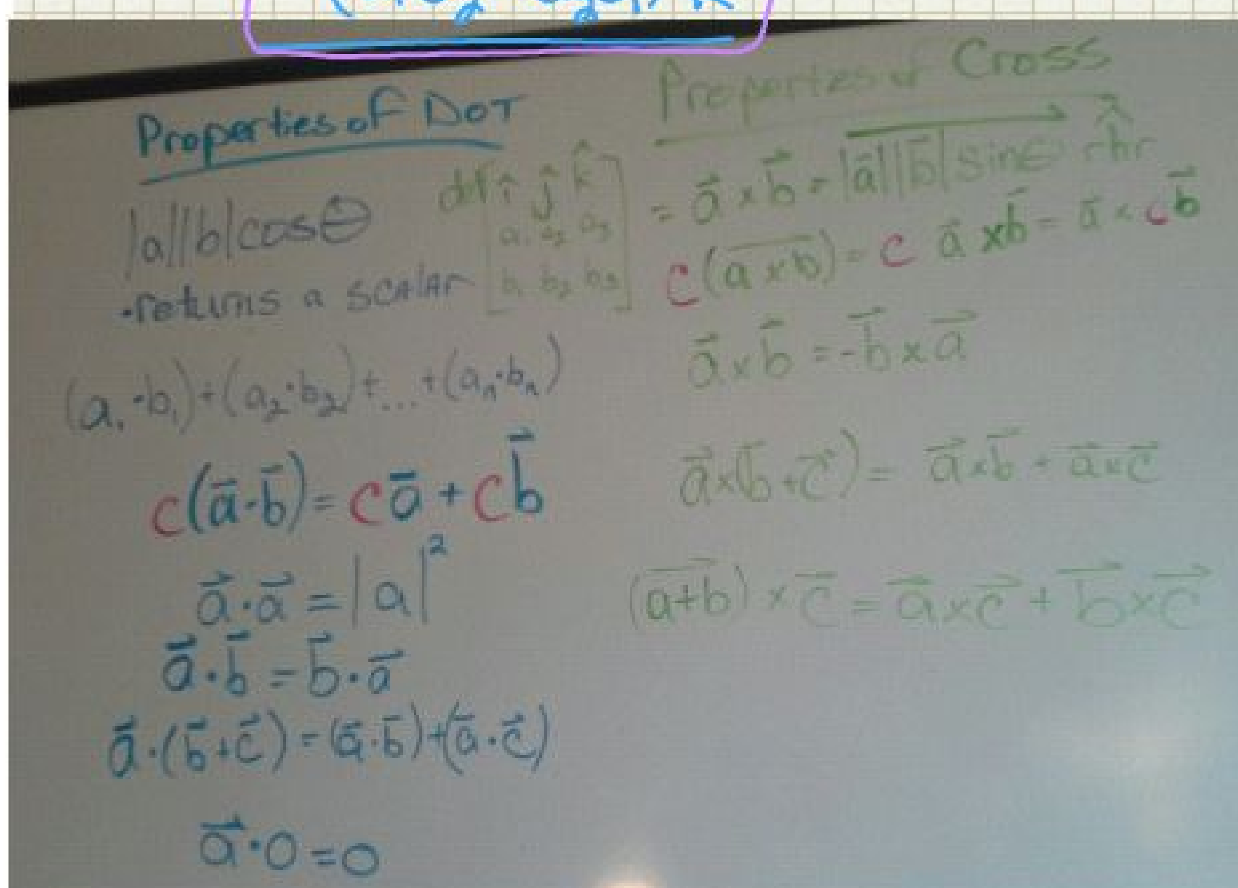
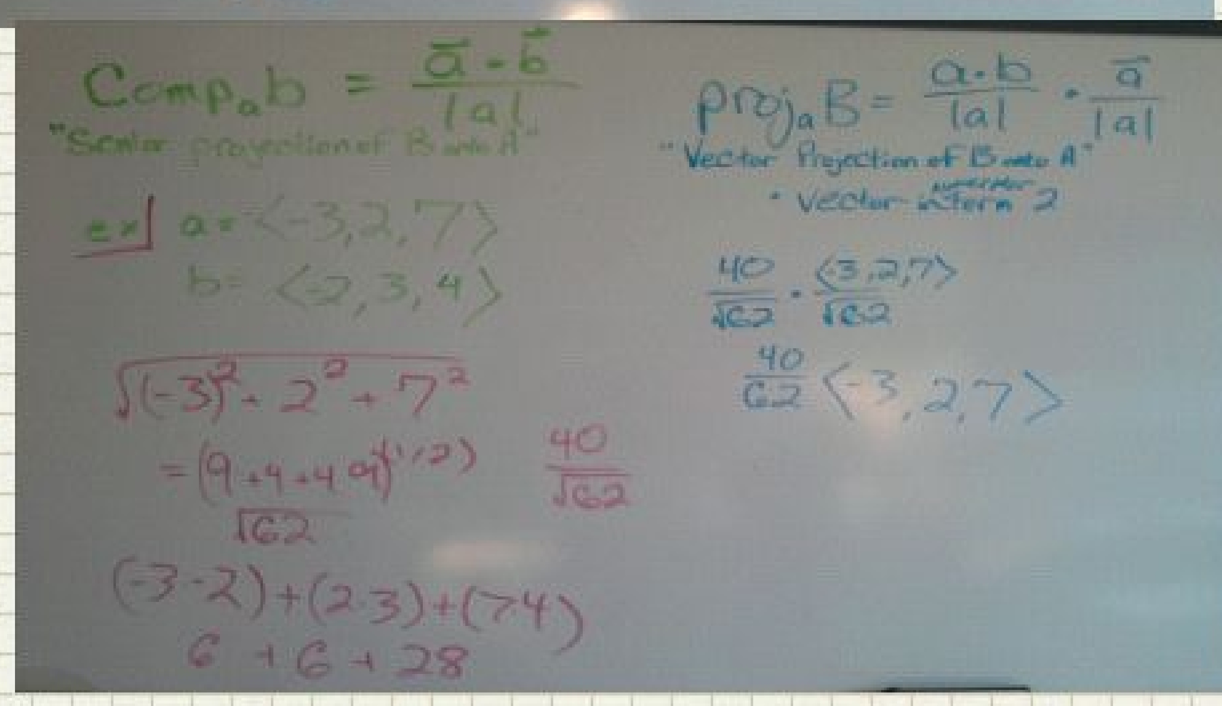


FIGURE 2





Area of $\overrightarrow{PQ} \times \overrightarrow{PR}$

$PQR = \frac{1}{2}|\overrightarrow{PQ} \times PR|$

Scalar Triple Product

returns a SCALAR

$V = \vec{a} \cdot (\vec{b} \times \vec{c})$

$\vec{a} = \langle a_1, a_2, a_3 \rangle$

$\vec{b} = \langle b_1, b_2, b_3 \rangle$

$\vec{c} = \langle c_1, c_2, c_3 \rangle$

$$\det\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$$

$(1, 0, 1)$ $(2, 3, 4)$

direction vector $v = \langle 1, 3, 3 \rangle$

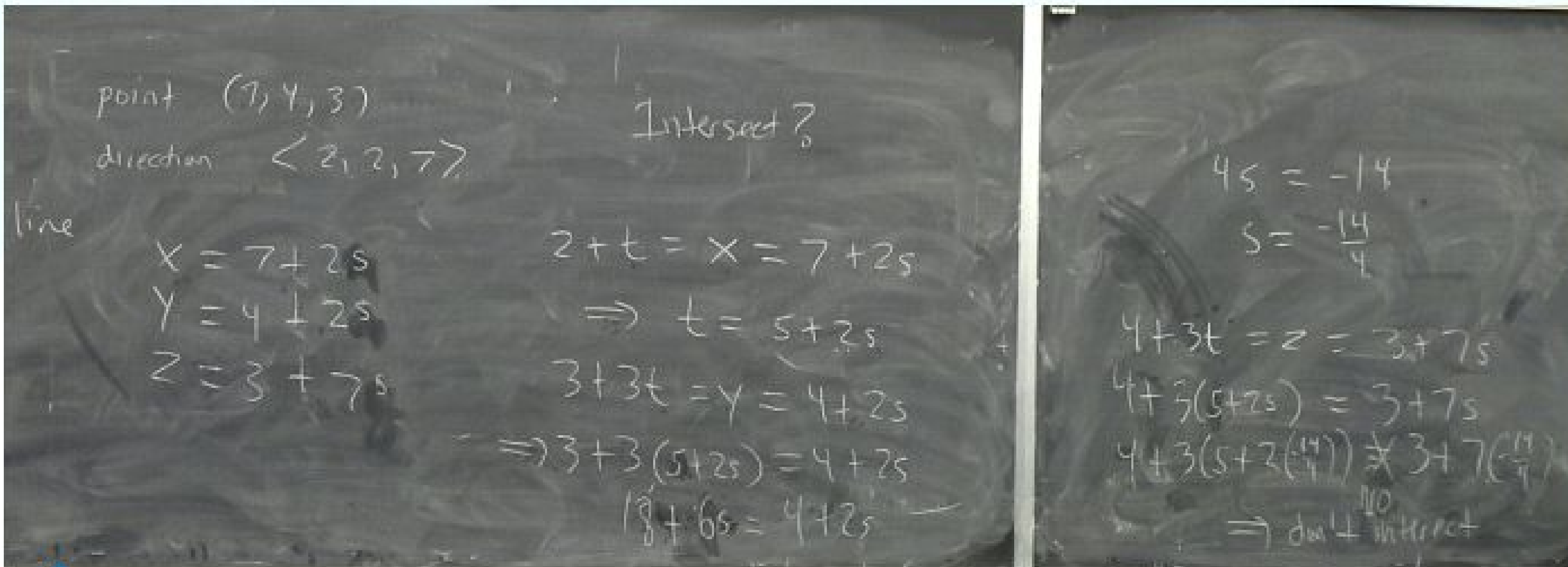
$$\left.\begin{aligned} x &= 2 + 1t \\ y &= 3 + 3t \\ z &= 4 + 3t \end{aligned}\right\}$$ Parametric equations (t is the parameter)

$$\underbrace{\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}}_{\text{symmetric equations}}$$

When does the line intersect the xy-plane? (z=0)

$z = 4 + 3t = 0 \quad x = 2 + (-4/3) = 2/3 \quad y = 3 + 3(-4/3) = -1$

$t = -4/3$, so $(2/3, -1, 0)$



Lines Point $(x_0, y_0, z_0) \rightarrow r_0 = \langle x_0, y_0, z_0 \rangle$ direction vector $V = \langle a, b, c \rangle$

$r = r_0 + tv$ {vector equation}

or

$$\left.\begin{aligned} x &= x_0 + ta \\ y &= y_0 + tb \\ z &= z_0 + tc \end{aligned}\right\}$$ Parametric Equations

or

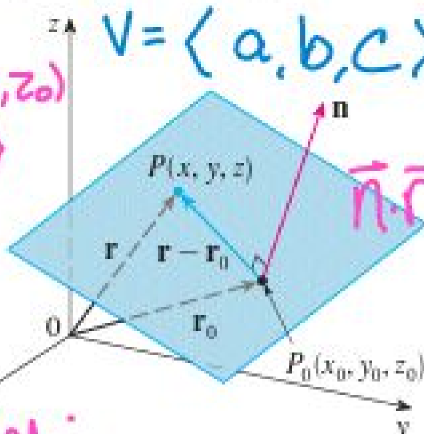
$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$ } symmetric equations

Point (x_0, y_0, z_0)

normal $= \langle a, b, c \rangle$

$r = \langle x, y, z \rangle$

$r_0 \langle x_0, y_0, z_0 \rangle$



$\vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{r}_0$

Planes:

$\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0$

$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

In general, the equation for a plane is ax+by+cz=0

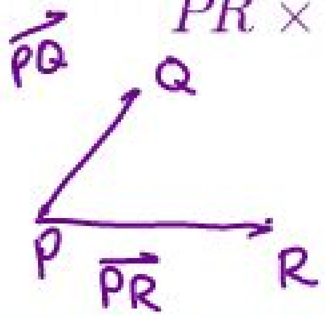
3 points determine a plane.

Point (1,3,9) Normal vector $\vec{n} = \langle 3, 5, 7 \rangle$

$3(x-1) + 5(y-3) + 7(z-9) = 0$

Create 2 vectors from three points with their differences. Cross the two vectors to create a third orthogonal vector that describes the plane

$P(1,3,2)Q(3,-1,6)R(5,2,0)$ $\quad \overrightarrow{PQ} = \langle 2,-4,4\rangle; \overrightarrow{PR} = \langle 4,-1,-2\rangle$



$$\overrightarrow{PR} \times \overrightarrow{PQ} = \begin{vmatrix} i & j & k \\ 4 & -1 & -2 \\ 2 & -4 & 4 \end{vmatrix}$$

$12i - 20j - 14k = n$

$12(x-1) - 20(y-3) - 14(z-2)$

Flat things in $\mathbb{R}^3$

Points

(x_0, y_0, z_0)

ex) $(2,1,-3)$

Lines

point (x_0, y_0, z_0)

direction vector (a,b,c)

$$\underbrace{x = x_0 + ta,\ y = y_0 + tb\ \ z = z_0 + tc}_{\text{Parametric Equations}}$$

-or-

$$\underbrace{\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}}_{\text{Symmetric equations}}$$

Planes

Point (x_0, y_0, z_0)

normal vector (a,b,c)

$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$

-or-

$ax + by + cz + d = 0$

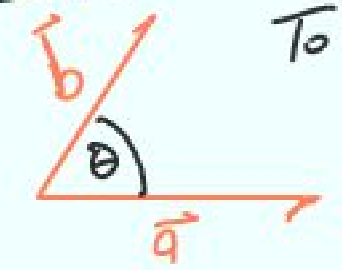
Intersect?

	Point	line	Plane
Point	SAme coordinates?	Plug in	Plug in
line		Ch. 9.5 ex 3	Ch. 9.5 ex 6
Plane			9.5 ex 7 b

Lines $x = 1 + 5t \quad y = -1 - 3t \quad z = 4 + 6t$ (use ParametricPlot3d)

Axeslabel → true

Planes $\vec{a}$ and $\vec{b}$ are vectors in a plane.



To parameterize a plane,

$u\vec{a} + v\vec{b}$ for all scalars $u + v$.

$4x + 5y - 2z - 5 = 0$

point $P_0(-2, 3, 1)$ is on plane

Another point?

$P_1(0, 0, -2.5) \longrightarrow \overrightarrow{P_0P_1} = \langle 2, -3, -3.5 \rangle = a$

$P_2(0, 1, 0) \qquad \overrightarrow{P_0P_2} = \langle 2, -2, -1 \rangle = b$

$\longrightarrow P_0 + ua + vb$ gives $\langle -2, 3, 1 \rangle + u\langle 2, -3, -3.5 \rangle + v\langle 2, -2, -1 \rangle$

$x = -2 + u2 + v2 \qquad y = 3 - u3 - v2 \qquad z = 1 - u3.5 - v$

Next: Curvey things

Definition A **function f of two variables** is a rule that assigns to each ordered pair of real numbers (x, y) in a set D a unique real number denoted by $f(x, y)$. The set D is the **domain** of f and its **range** is the set of values that f takes on, that is, $\{f(x, y) \mid (x, y) \in D\}$.

Definition If f is a function of two variables with domain D, then the **graph** of f is the set of all points (x, y, z) in $\mathbb{R}^3$ such that $z = f(x, y)$ and (x, y) is in D.

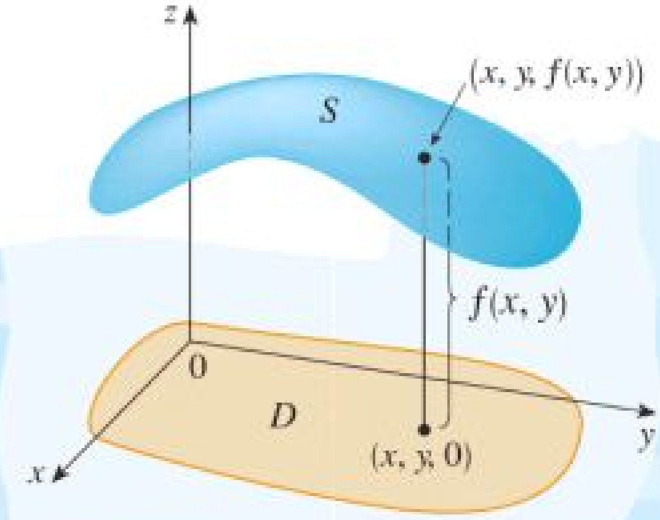
Conic Sections

Ellipse

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

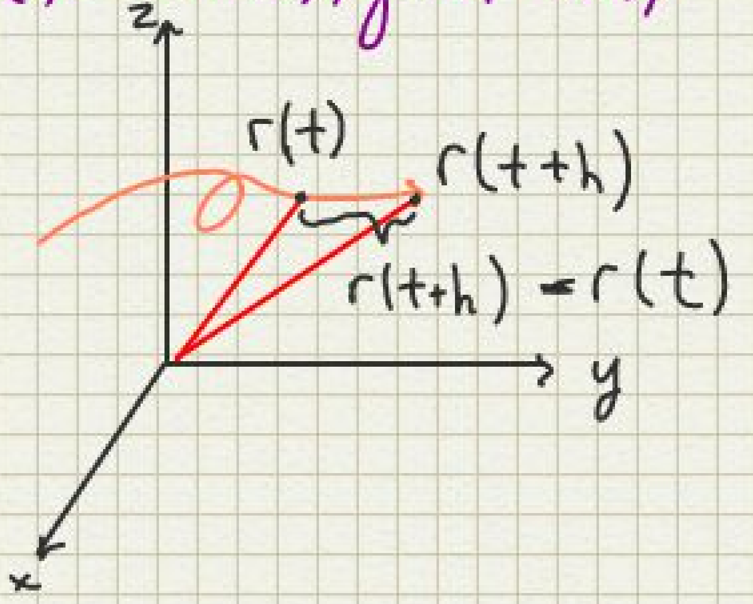
Hyperbola

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

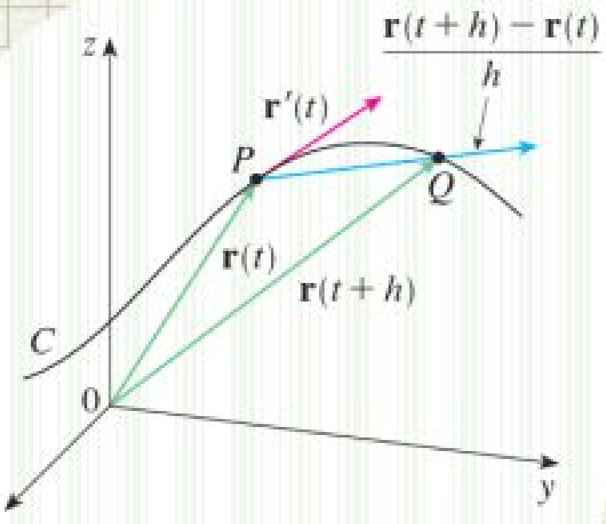


Space Curve

$r(t) = \langle f(t), g(t), h(t) \rangle$



$$\frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = \lim_{h \to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}$$



(b) The tangent vector

2 Theorem If $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$, where f, g, and h are differentiable functions, then

$$\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle = f'(t)\,\mathbf{i} + g'(t)\,\mathbf{j} + h'(t)\,\mathbf{k}$$

3 Theorem Suppose $\mathbf{u}$ and $\mathbf{v}$ are differentiable vector functions, c is a scalar, and f is a real-valued function. Then

1. $\frac{d}{dt}[\mathbf{u}(t) + \mathbf{v}(t)] = \mathbf{u}'(t) + \mathbf{v}'(t)$

2. $\frac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$

3. $\frac{d}{dt}[f(t)\mathbf{u}(t)] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$

4. $\frac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)] = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$

5. $\frac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)] = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$

6. $\frac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\mathbf{u}'(f(t))$ (Chain Rule)

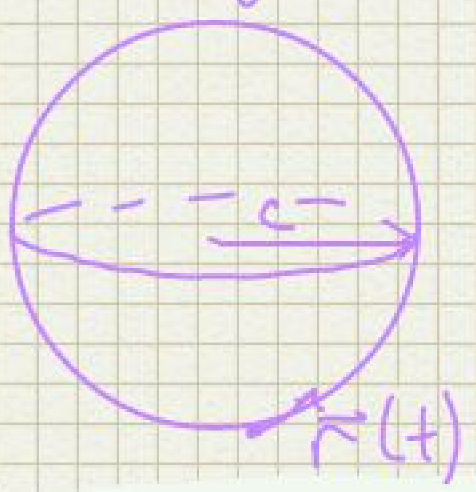
$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$

$\vec{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle$

$\frac{d}{dt}\, \vec{r}(t) \cdot \vec{r}(t) = |r(t)^2| = c^2$

$\frac{d}{dt}\, \vec{r}(t) \cdot \vec{r}'(t) = 0$

Note If $|r(t)| = c$, r lies on a sphere of radius c



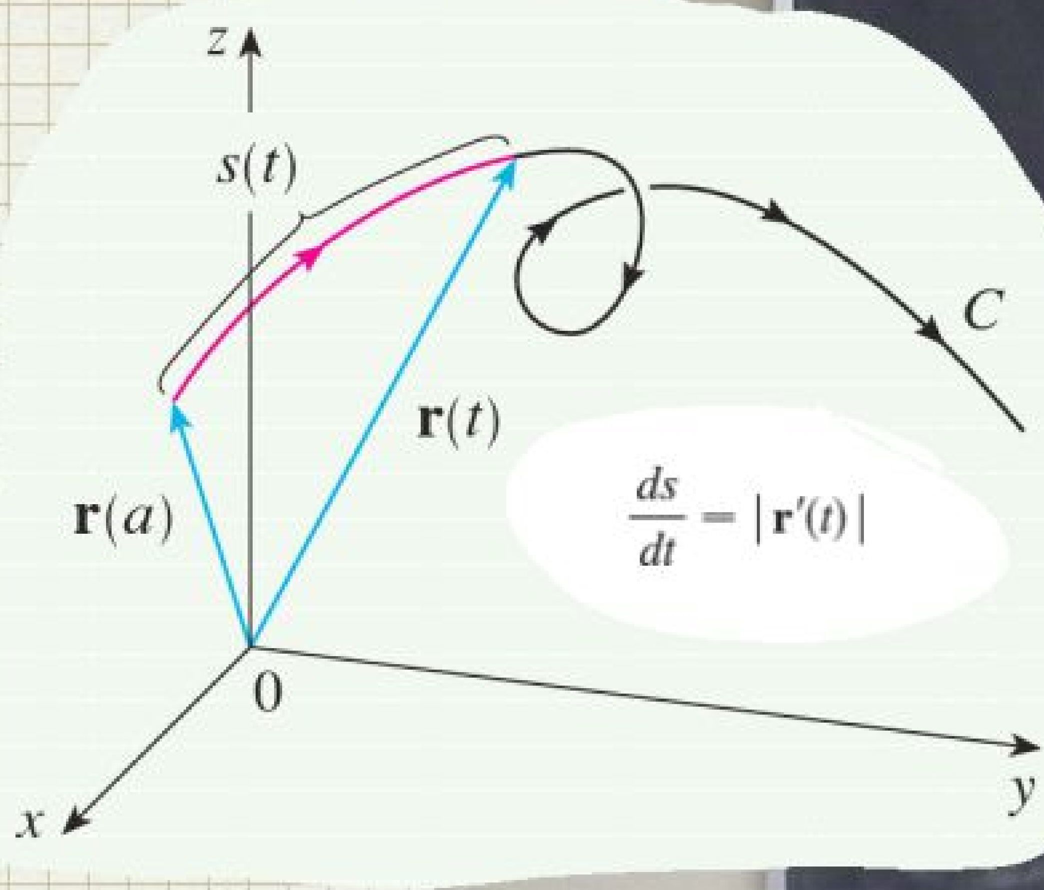
$\Rightarrow \vec{r}'(t)\vec{r}(t) + \vec{r}(t)\, r'(t)$

$= 2\vec{r}(t)\vec{r}'(t) = 0$

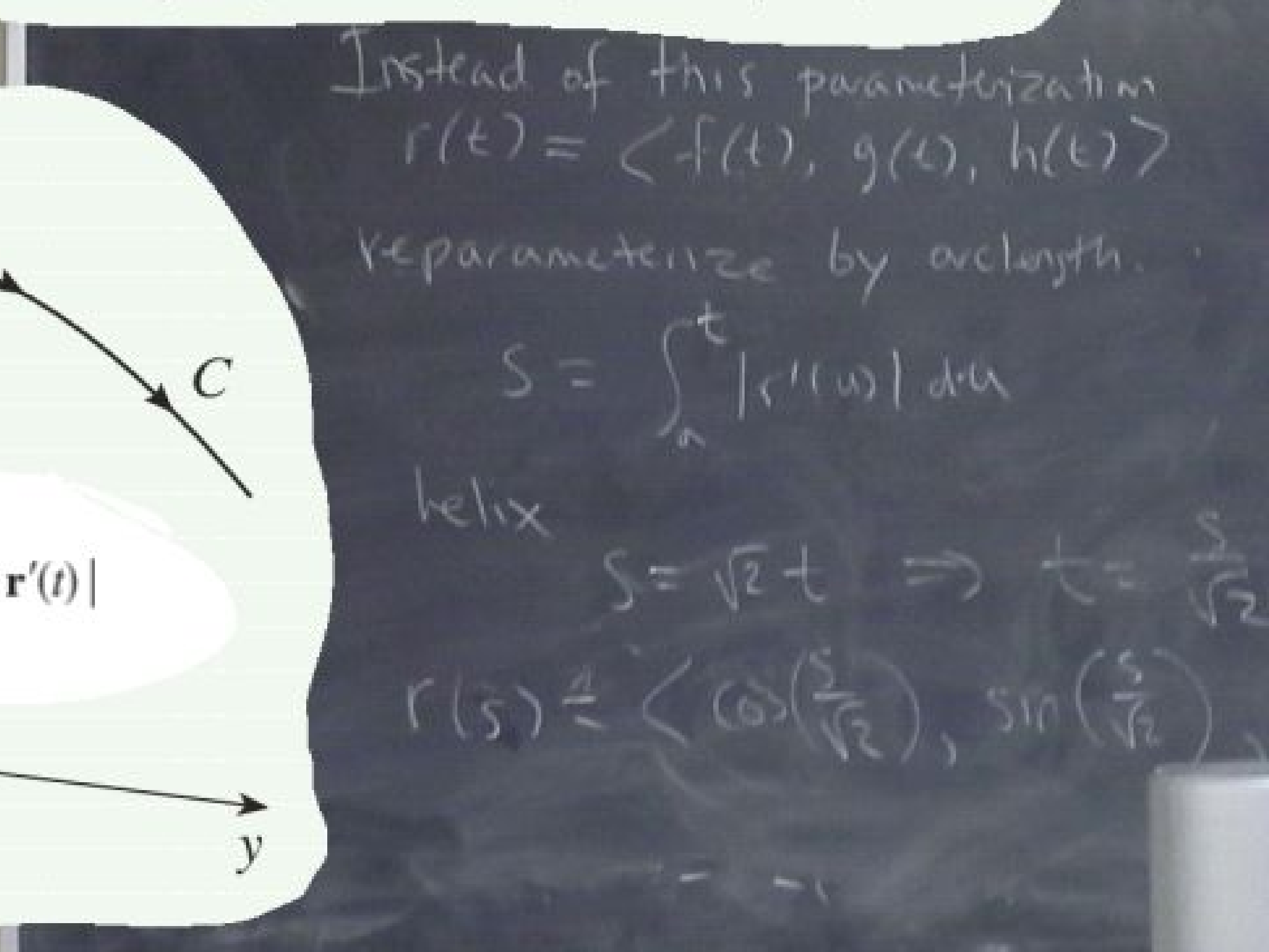
Arc Length in 3d:

$$\boxed{L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\, dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt}$$

$$s(t) = \int_a^t |\mathbf{r}'(u)|\, du = \int_a^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2}\, du$$



$$\frac{ds}{dt} = |\mathbf{r}'(t)|$$



8 Definition The **curvature** of a curve is

$$\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} \qquad \kappa = \left|\frac{d\mathbf{T}}{ds}\right| \qquad \kappa(x) = \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}}$$

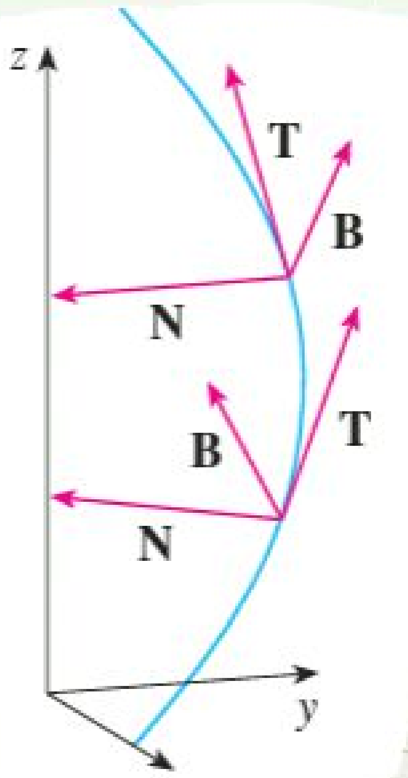
where **T** is the unit tangent vector.

10 Theorem The curvature of the curve given by the vector function **r** is

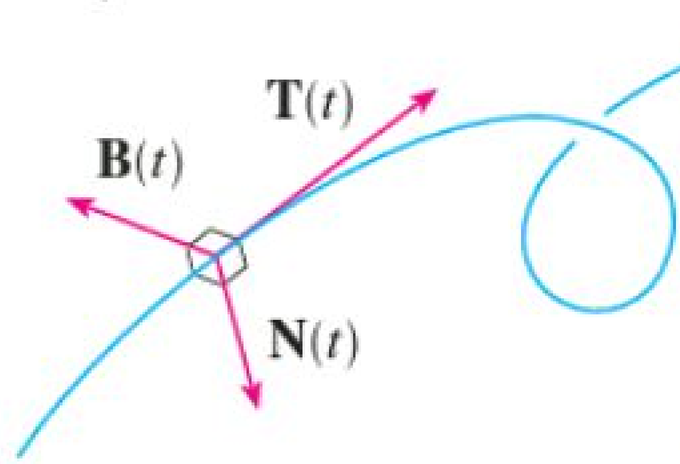
$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} \qquad \mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} \qquad \mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$$

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$



We can think of the normal vector as indicating the direction in which the curve is turning at each point.



Fernet Frames

HW

10.3 ; 19 + moArs

21, 25, 37, 38, 39, 40

10.5

~~1,3,4,5,6,7,11,12~~,13

14, 15, 16, 17, 19, 20,

27

May 2, 2013

Ch. 11.1

1, 5, 9, 10, 13, 14, 15, 16, 17, 18

29-38 all

Ch 11.2:

5, 6, 7, 8, 11, 12, 15, 19

20, 25, 28

11.3)

7, 8, 13, 14, 15

16, 23, 27, 35, 36, 41, 42, 47, 48, 55, 63

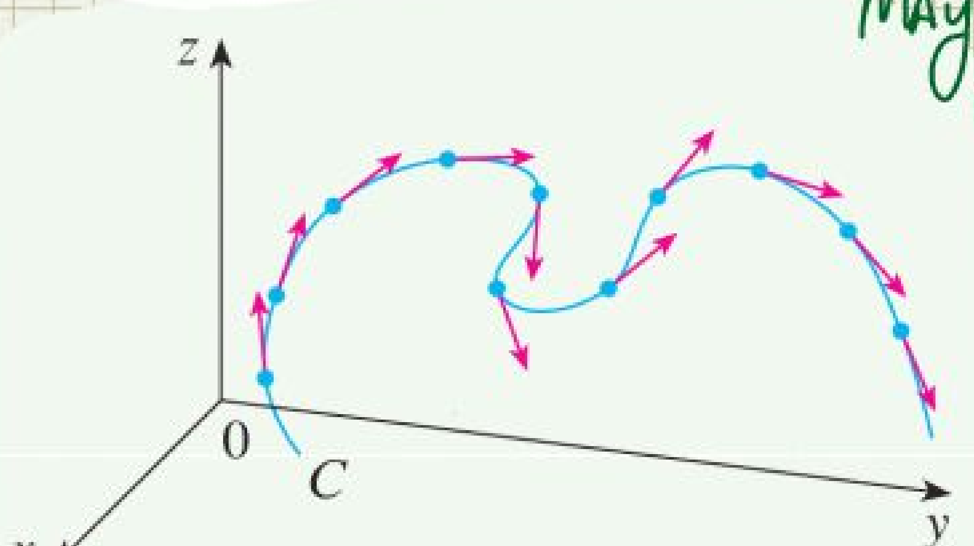
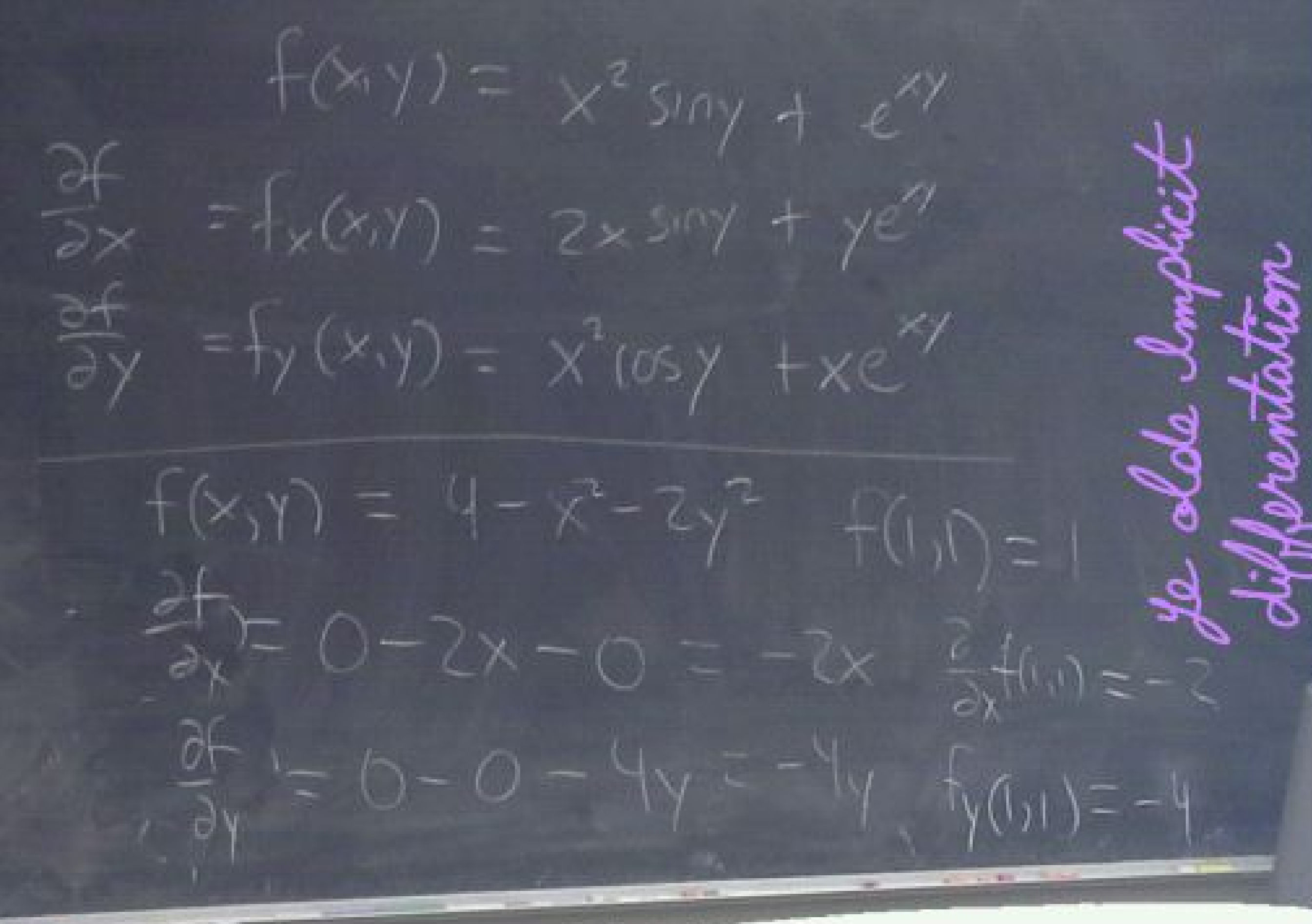


FIGURE 4

Unit tangent vectors at equally spaced



ye olde Implicit differentiation

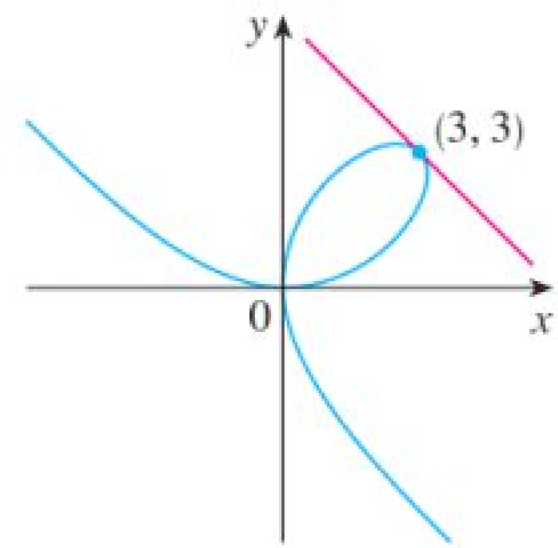
$$3x^2 + 3y^2y' = 6xy' + 6y$$

$$x^2 + y^2y' = 2xy' + 2y$$

$$y^2y' - 2xy' = 2y - x^2$$

$$(y^2 - 2x)y' = 2y - x^2$$

$$y' = \frac{2y - x^2}{y^2 - 2x}$$



Rule for Finding Partial Derivatives of $z = f(x, y)$

1. To find f_x, regard y as a constant and differentiate $f(x, y)$ with respect to x.
2. To find f_y, regard x as a constant and differentiate $f(x, y)$ with respect to y.

V EXAMPLE 4 Implicit partial differentiation Find $\partial z/\partial x$ and $\partial z/\partial y$ if z is defined implicitly as a function of x and y by the equation

$$x^3 + y^3 + z^3 + 6xyz = 1$$

SOLUTION To find $\partial z/\partial x$, we differentiate implicitly with respect to x, being careful to treat y as a constant:

$$3x^2 + 3z^2\frac{\partial z}{\partial x} + 6yz + 6xy\frac{\partial z}{\partial x} = 0$$

Solving this equation for $\partial z/\partial x$, we obtain

$$\frac{\partial z}{\partial x} = -\frac{x^2 + 2yz}{z^2 + 2xy}$$

Similarly, implicit differentiation with respect to y gives

$$\frac{\partial z}{\partial y} = -\frac{y^2 + 2xz}{z^2 + 2xy}$$

11.4

Point (x_0, y_0, z_0)

Plane

$$A(x-x_0)+B(y-y_0)+C(z-z_0)=0$$

$$z-z_0 = -\frac{A}{C}(x-x_0) - \frac{B}{C}(y-y_0)$$

$$a = -\frac{A}{C} \quad \text{and} \quad b = -\frac{B}{C}$$

$$z-z_0 = a(x-x_0)+b(y-y_0)$$

If $x=x_0$ this

$z-z_0 = b(y-y_0)$ (point-slope form of line)

So b is the slope of tangent line $= \frac{\partial f}{\partial y}$

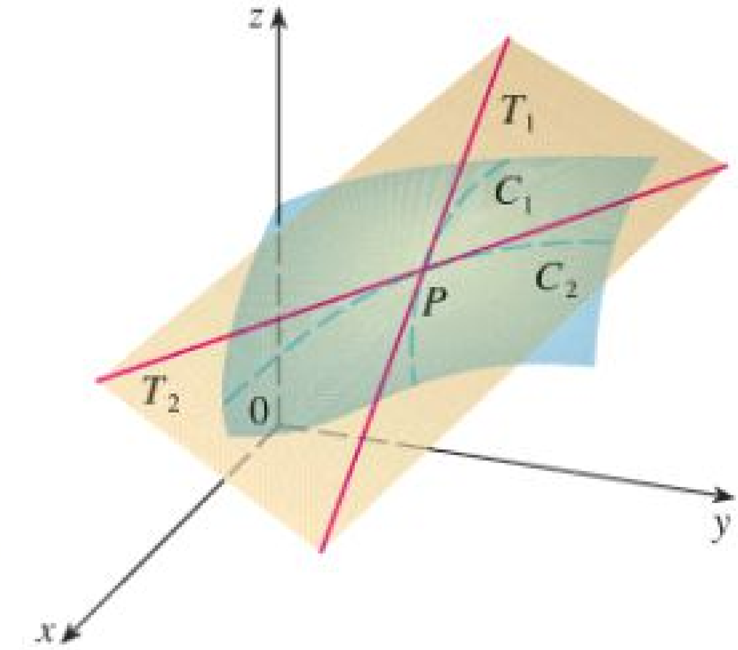


FIGURE 1

The tangent plane contains the tangent lines T_1 and T_2.

$$A(x-x_0)+B(y-y_0)+C(z-z_0)=0$$

By dividing this equation by C and letting $a=-A/C$ and $b=-B/C$, we can write it in the form

$$\boxed{1} \quad z - z_0 = a(x-x_0)+b(y-y_0)$$

$\boxed{2}$ Suppose f has continuous partial derivatives. An equation of the tangent plane to the surface $z=f(x, y)$ at the point $P(x_0, y_0, z_0)$ is

$$z - z_0 = f_x(x_0, y_0)(x-x_0) + f_y(x_0, y_0)(y-y_0)$$

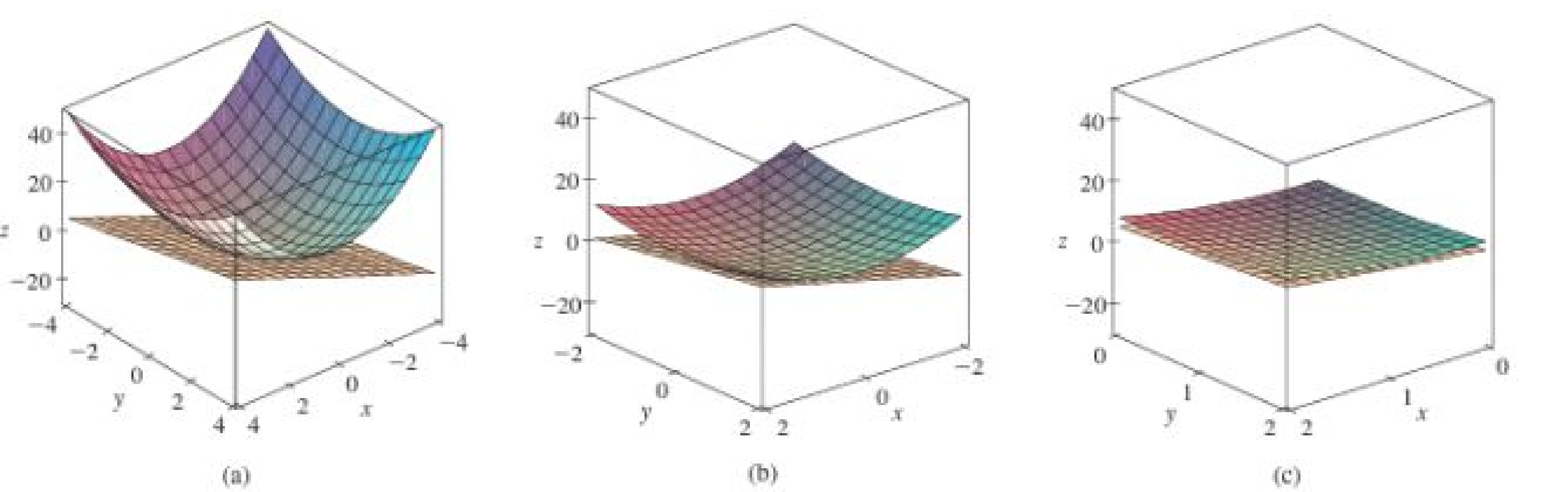
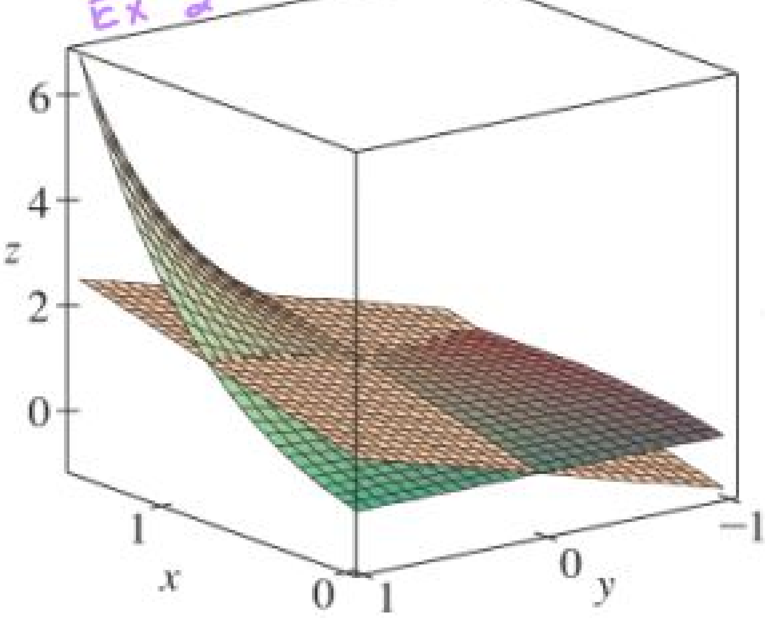


FIGURE 2 The elliptic paraboloid $z=2x^2+y^2$ appears to coincide with its tangent plane as we zoom in toward $(1, 1, 3)$.

Ex 2



V EXAMPLE 2 **Using a linearization to estimate a function value**

Show that $f(x, y) = xe^{xy}$ is differentiable at $(1, 0)$ and find its linearization there. Then use it to approximate $f(1.1, -0.1)$.

SOLUTION The partial derivatives are

$$f_x(x, y) = e^{xy} + xye^{xy} \qquad f_y(x, y) = x^2e^{xy}$$

$$f_x(1, 0) = 1 \qquad f_y(1, 0) = 1$$

Both f_x and f_y are continuous functions, so f is differentiable by Theorem 8. The linearization is

$$L(x, y) = f(1, 0) + f_x(1, 0)(x-1) + f_y(1, 0)(y-0)$$

$$= 1 + 1(x-1) + 1 \cdot y = x + y$$

The corresponding linear approximation is

$$xe^{xy} \approx x + y$$

so

$$f(1.1, -0.1) \approx 1.1 - 0.1 = 1$$

Compare this with the actual value of $f(1.1, -0.1) = 1.1e^{-0.11} \approx 0.98542$.

1, 2, 3, 5, 9, 11, 12

13, 19, 21, 22, 42

Chain rule 11.5 $f(g(t))=y$

$$t \longrightarrow \boxed{g} \xrightarrow{x} \boxed{f} \longrightarrow y \qquad \frac{dy}{dt}=\frac{dy}{dx}\frac{dx}{dt}$$

As $\Delta t \to 0$, what happens?

$$\lim_{\Delta t\to 0}\frac{\Delta x}{\Delta t}=\frac{dx}{dt}, \quad \lim_{\Delta t\to 0}\frac{\Delta y}{\Delta t}=\frac{dy}{dt}$$

$$\begin{tikzcd} & \boxed{g} \arrow[dr, "x"] & \\ t \arrow[ur] \arrow[dr] & & \boxed{f} \arrow[r] & z \\ & \boxed{h} \arrow[ur, "y"'] & \end{tikzcd}$$

assume f, g, h are differentiable

$$f(x,y)=f(g(t),h(t))=z$$

Show $\dfrac{dz}{dt}=\dfrac{\partial f}{\partial x}\dfrac{dx}{dt}+\dfrac{\partial f}{\partial y}\dfrac{dy}{dt}$

Recall f is differentiable

$$\Delta z = f_x(a,b)\Delta x + f_y(a,b)\Delta y + \varepsilon_1\Delta x + \varepsilon_2\Delta y$$

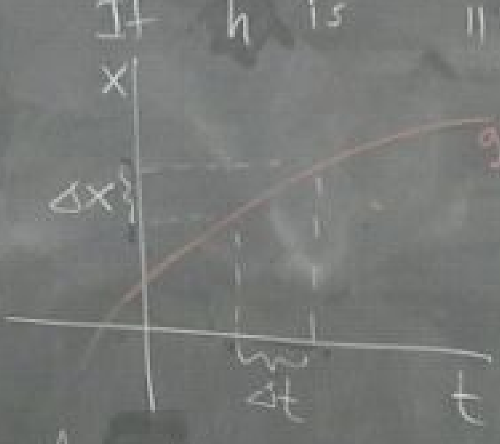
if $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y)\to(0,0)$

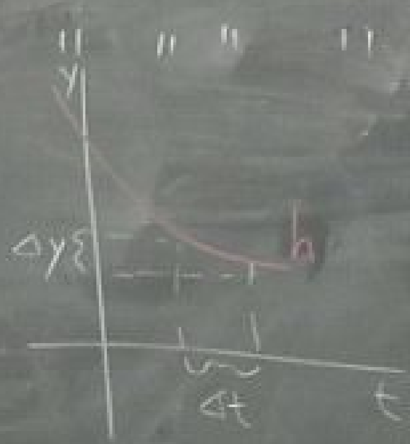
Recall $\dfrac{dz}{dt}=\lim\limits_{\Delta t\to 0}\dfrac{\Delta z}{\Delta t}$

So $$\frac{\Delta z}{\Delta t}=f_x(a,b)\frac{\Delta x}{\Delta t}+f_y(a,b)\frac{\Delta y}{\Delta t}+\varepsilon_1\frac{\Delta x}{\Delta t}+\varepsilon_2\frac{\Delta y}{\Delta t}$$

$x=g(t)$ $\quad y=h(t)$

If g is differentiable then it is continuous
If h is " " " "





As $\Delta t \to 0$, we have $\Delta x \to 0$ So $\varepsilon_1 \to 0$

As $\Delta t \to 0$ we have $\Delta y \to 0$ So $\varepsilon_2 \to 0$

So $$\frac{dz}{dt}=\lim_{\Delta t\to 0}\frac{\Delta z}{\Delta t}$$

$$=\lim_{\Delta t\to 0}\left(\frac{\partial f}{\partial x}\frac{\Delta x}{\Delta t}+\frac{\partial g}{\partial y}\frac{\Delta y}{\Delta t}+\varepsilon_1\frac{\Delta x}{\Delta t}+\varepsilon_2\frac{\Delta y}{\Delta t}\right)$$

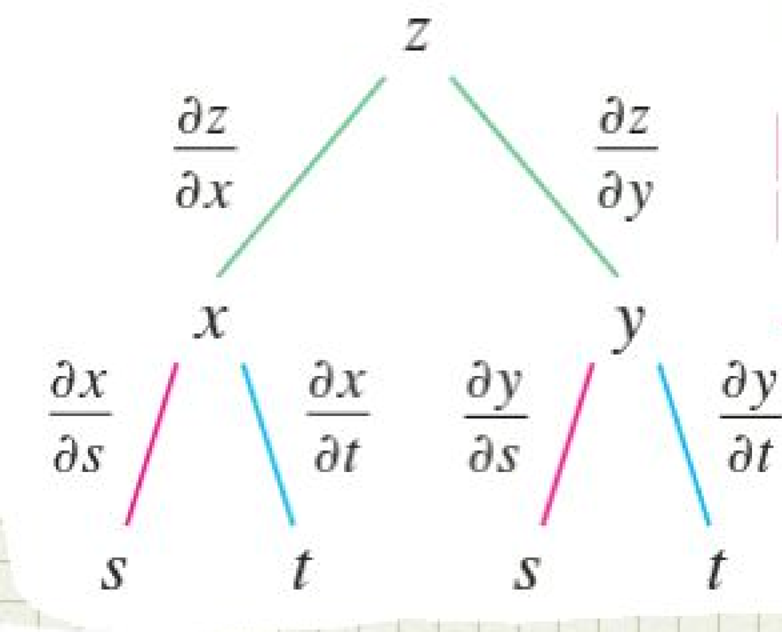
$$=\frac{\partial f}{\partial x}\frac{dx}{dt}+\frac{\partial g}{\partial y}\frac{dy}{dt}+0\cdot(\quad)+0\cdot(\quad)$$

$$=\frac{\partial f}{\partial x}\frac{dx}{dt}+\frac{\partial g}{\partial y}\frac{dy}{dt}$$

$$\begin{tikzcd} & z \arrow[dl, "\frac{\partial z}{\partial x}"'] \arrow[dr, "\frac{\partial z}{\partial y}"] & \\ x \arrow[d, "\frac{dx}{dt}"'] & & y \arrow[d, "\frac{dy}{dt}"] \\ t & & t \end{tikzcd}$$

2 The Chain Rule (Case 1) Suppose that $z=f(x,y)$ is a differentiable function of x and y, where $x=g(t)$ and $y=h(t)$ are both differentiable functions of t. Then z is a differentiable function of t and

$$\frac{dz}{dt}=\frac{\partial f}{\partial x}\frac{dx}{dt}+\frac{\partial f}{\partial y}\frac{dy}{dt}$$



3 The Chain Rule (Case 2) Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(s, t)$ and $y = h(s, t)$ are differentiable functions of s and t. Then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

Example: $z = f(x,y) = e^x \sin y$

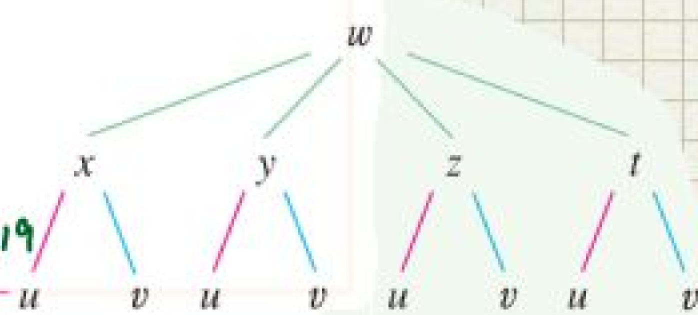
$x = st^2 \quad y = s^2 t$

$$\therefore \frac{dz}{dt} = (e^x \sin y)(2st) + (e^x \cos y)(s^2)$$

4 The Chain Rule (General Version) Suppose that u is a differentiable function of the n variables $x_1, x_2, \ldots, x_n$ and each x_j is a differentiable function of the m variables $t_1, t_2, \ldots, t_m$. Then u is a function of $t_1, t_2, \ldots, t_m$ and

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

for each $i = 1, 2, \ldots, m$.

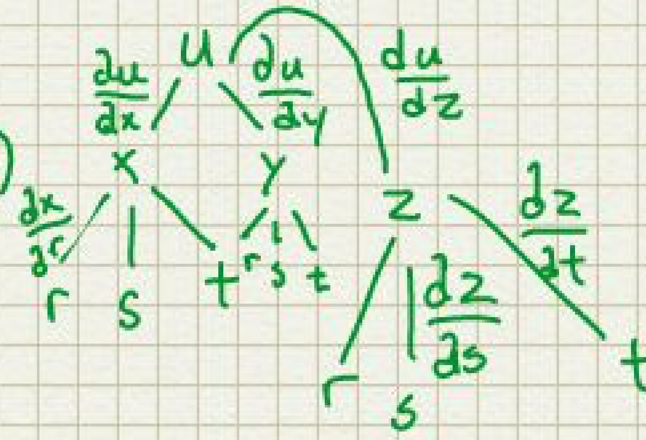


HW 11.5 = 1, 2, 3, 5, 6, 7, 8, 11, 13, 14, 17, 19, 20, 22, 23, 25, 26

for $f(x,y,z) = x^4 y + y^2 z = u$

and $x = rse^t$, $y = rs^2e^{-t}$, $z = r^2 s(\sin t)$

$$\frac{du}{dr} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial r}$$



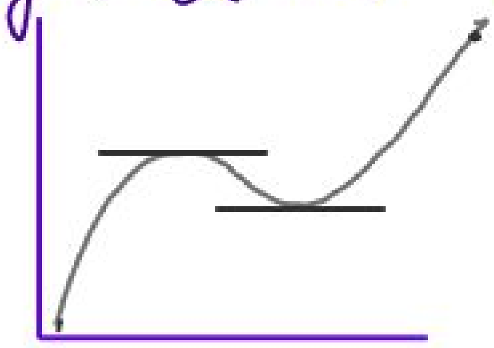
Tangent Planes Ex) $\frac{x^2}{4}+y^2+\frac{z^2}{9}=3$ $f_x=\frac{x}{2}$ $f_y=2y$ $f_z=\frac{2z}{9}$

at $(-2,1,-3)$ $f_x(-2)=-1$, $f_y(1)=2$, $f_z(-3)=\frac{-2}{3}$

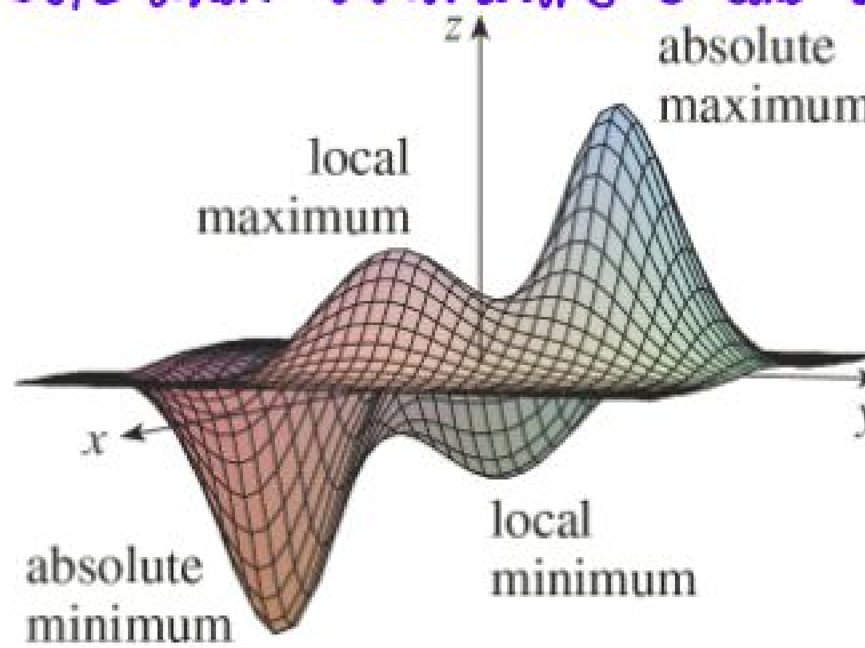
so a tangent plane is

$-1(x+2)+2(y-1)-\frac{2}{3}(z+3)$

Optimization



Check when derivative = 0 and endpoints



Midterm Prep

1 Definition A function of two variables has a **local maximum** at (a, b) if $f(x, y) \leqslant f(a, b)$ when (x, y) is near (a, b). [This means that $f(x, y) \leqslant f(a, b)$ for all points (x, y) in some disk with center (a, b).] The number $f(a, b)$ is called a **local maximum value**. If $f(x, y) \geqslant f(a, b)$ when (x, y) is near (a, b), then f has a **local minimum** at (a, b) and $f(a, b)$ is a **local minimum value**.

2 Fermat's Theorem for Functions of Two Variables If f has a local maximum or minimum at (a, b) and the first-order partial derivatives of f exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

Ch. 9

dot + cross product

use $|a||b|\cos\theta$ to find θ

→ Equation of a line/plane

line of intersection

cylindrical + spherical coords

- curvature
- arc length
- intersection of surfaces
- Parameterize a Plane
- Partial + Second Partials
- Chain Rule

- Exam

basic vector stuff -distance ✓
-eq of sphere
-addition stuff ✓
→ dot products ✓ → definition $|a||b|\cos\theta|$
→ x products ✓ → def $||a||b|\sin\theta|$ a

equations of lines and planes →
→ line point direct vecs
→ difference btwn line and plane

functions of 2 variables
→ cylindrical spherical coordinates ✓✓

space curves
→ arc length
→ curvature
→ find parametric eq for intersection btwn 2 surfaces

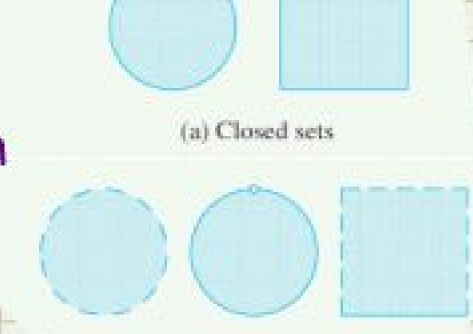
surfaces
→ parametric eq for a plane using 2 vectors + point

HW: 11.7) 1,5,6,7,11,13
17,19,21,27,28
43,44

partial derivative
→ 1^{st} 2^{nd}
→ chain rule

27–32 Find the absolute maximum and minimum values of f on the set D. Extreme Value Theorem in 3d, restrict to a bounded region

27. $f(x, y) = 1 + 4x - 5y$, D is the closed triangular region with vertices $(0, 0)$, $(2, 0)$, and $(0, 3)$



(a) Closed sets

(b) Sets that are not closed

8 **Extreme Value Theorem for Functions of Two Variables** If f is continuous on a closed, bounded set D in $\mathbb{R}^2$, then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in D.

43. Find the volume of the largest rectangular box in the first octant with three faces in the coordinate planes and one vertex in the plane $x + 2y + 3z = 6$.

Maximize Volume $V = xyz$ subject to the constraint $x+2y+3z=6$. $z = \dfrac{6-2y-x}{3}$

$V(x,y) = xy\left(\dfrac{6-2y-x}{3}\right)$

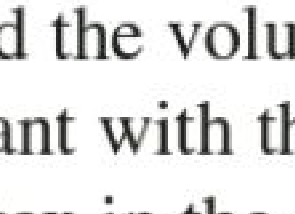
Take partials set to zero & solve

Plug values & find extrema

44. Find the dimensions of the rectangular box with largest volume if the total surface area is given as 64 cm^2.

$V(x,y) = xy\left(\dfrac{32-xy}{y+x}\right)$

$V = xyz$



6 sides $\rightarrow 2xy + 2xz + 2yz = 64$

$z = \dfrac{64-2xy}{2y+2x} = \dfrac{32-xy}{y+x}$

$V_x(x,y) \Rightarrow$ set to $\emptyset$

$V_y(x,y) \rightarrow$ set zero

Friday May 31, '13

$f(x,y,z) = k$ · constraint · Forms level surface

If (x_0, y_0, z_0) gives a maximum then consider $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ going through (x_0, y_0, z_0) when $t = t_0$.

Then $\dfrac{\partial r}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial r}{\partial y}\dfrac{dy}{dt} + \dfrac{\partial r}{\partial z}\dfrac{dz}{dt} = 0$

$\nabla f(x_0, y_0, z_0) \cdot \vec{r}(t) = 0$

Method of Lagrange Multipliers To find the maximum and minimum values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$ [assuming that these extreme values exist and $\nabla g \neq \mathbf{0}$ on the surface $g(x, y, z) = k$]:

(a) Find all values of x, y, z, and λ such that

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$$

and $$g(x, y, z) = k$$

(b) Evaluate f at all the points (x, y, z) that result from step (a). The largest of these values is the maximum value of f; the smallest is the minimum value of f.

11.8 Lagrange Multipliers HW: 3,5,7,8,11,12,15,21

11. $f(x, y, z) = x^2 + y^2 + z^2; \quad x^4 + y^4 + z^4 = 1$

$\nabla f = \lambda \nabla G \rightarrow \nabla f = \langle 2x, 2y, 2z \rangle ; \nabla g = \langle 4x^3, 4y^3, 4z^3 \rangle$

$\therefore \; 2x = \lambda 4x^3, \; 2y = \lambda 4y^3, \; 2z = \lambda 4z^3$

$x = \lambda 2x^3 \qquad \lambda = \frac{1}{2x^2}$

$\frac{x}{2} = \lambda x^3$

21. Consider the problem of minimizing the function $f(x, y) = x$ on the curve $y^2 + x^4 - x^3 = 0$ (a piriform).

(a) Try using Lagrange multipliers to solve the problem.

(b) Show that the minimum value is $f(0, 0) = 0$ but the Lagrange condition $\nabla f(0, 0) = \lambda \nabla g(0, 0)$ is not satisfied for any value of λ.

(c) Explain why Lagrange multipliers fail to find the minimum value in this case.

a) $\nabla f = \langle 1, 0 \rangle ; \nabla g = \langle 4x^3 - 3x^2, 2y \rangle$

$\nabla f = \lambda \nabla g \qquad 1 = \lambda(4x^3 - 3x^2) ; \; 0 = \lambda 2y \quad \therefore 0 = x^4 - x^3$

$x \neq 0 \qquad \lambda \neq 0, \; y = 0 \qquad 0 = x^3(x-1)$

$x = 1 \therefore \lambda = 1$

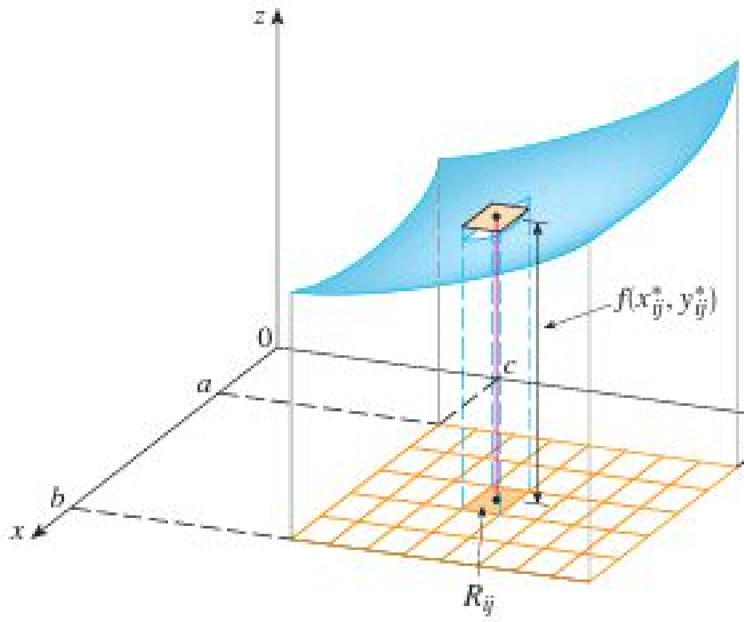


FIGURE 4

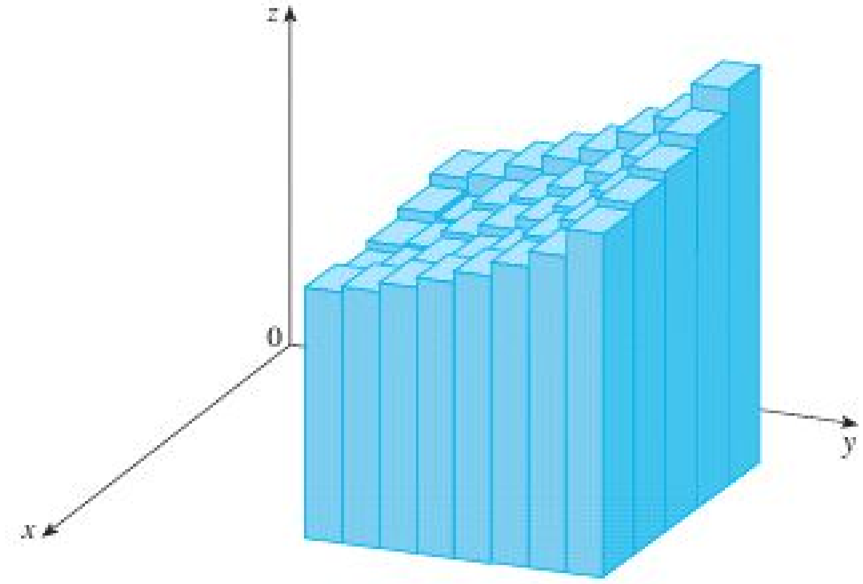


FIGURE 5

Chapter 12

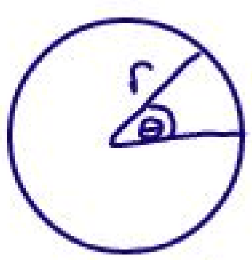
12.1 3, 4, 9, 10, 11, 12

12.2 3, 5, 6, 9, 10, 13, 14, 17, 21
23, 24. 25

5 Definition The **double integral** of f over the rectangle R is

$$\iint_R f(x, y)\, dA = \lim_{m, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\, \Delta A$$

if this limit exists.



Area of slice

$\frac{\theta}{2\pi}(\pi r^2)$

$= \frac{\theta}{2} r^2$

3 If f is continuous on a type I region D such that

$$D = \{(x, y) \mid a \leqslant x \leqslant b,\ g_1(x) \leqslant y \leqslant g_2(x)\}$$

then

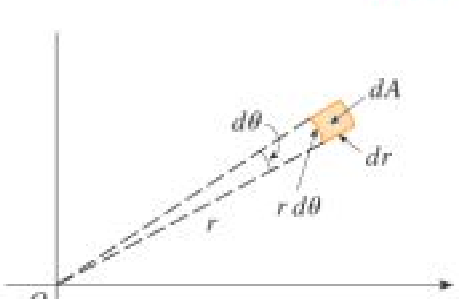
$$\iint_D f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx$$

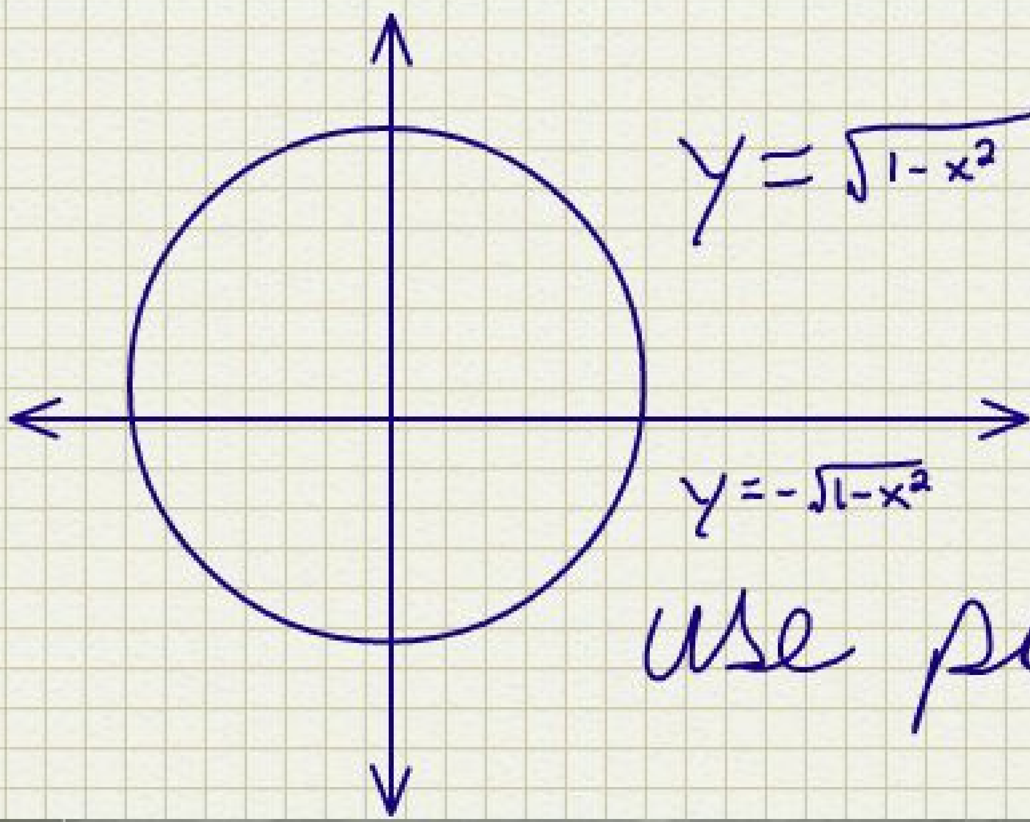
$$r^2 = x^2 + y^2 \qquad x = r\cos\theta \qquad y = r\sin\theta$$

2 Change to Polar Coordinates in a Double Integral If f is continuous on a polar rectangle R given by $0 \leqslant a \leqslant r \leqslant b$, $\alpha \leqslant \theta \leqslant \beta$, where $0 \leqslant \beta - \alpha \leqslant 2\pi$, then

$$\iint_R f(x, y)\, dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

The formula in (2) says that we convert from rectangular to polar coordinates in a double integral by writing $x = r\cos\theta$ and $y = r\sin\theta$, using the appropriate limits of integration for r and θ, and replacing dA by $r\,dr\,d\theta$. Be careful not to forget the additional factor r on the right side of Formula 2. A classical method for remembering this is shown





$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}$$

$\uparrow$

NO

use polar

Ex $f(x,y) = 1 - x^2 - y^2$ $\Rightarrow x^2 + y^2 = 1$

In the xy-plane $1 - x^2 - y^2 = 0$

$$D = \{(r,\theta) \mid 0 \le r \le 1,\ 0 \le \theta \le 2\pi\}$$

$$\iint_D 1 - x^2 - y^2 \, dA$$

In polar

$$= \int_0^{2\pi}\left[\int_0^1 \left(1 - (r\cos\theta)^2 - (r\sin\theta)^2\right) r \, dr\right] d\theta$$

$$\int_0^{2\pi}\int_0^1 1 - r^2(\cos^2\theta + \sin^2\theta)\, r \, dr \, d\theta$$

$$\int_0^{2\pi}\int_0^1 (1 - r^2)\, r \, dr \, d\theta$$

$$= \int_0^{2\pi}\int_0^1 r - r^3 \, dr \, d\theta$$

$$= \int_0^{2\pi} \left.\frac{r^2}{2} - \frac{r^4}{4}\right]_{r=0}^{r=1} d\theta$$

$$\int_0^{2\pi} \frac{1}{2} - \frac{1}{4} \, d\theta$$

$$\int_0^{2\pi} \frac{1}{4} \, d\theta$$

$$\left.\frac{1}{4}\theta\right]_0^{2\pi} = \frac{1}{4}2\pi = \frac{\pi}{2}$$

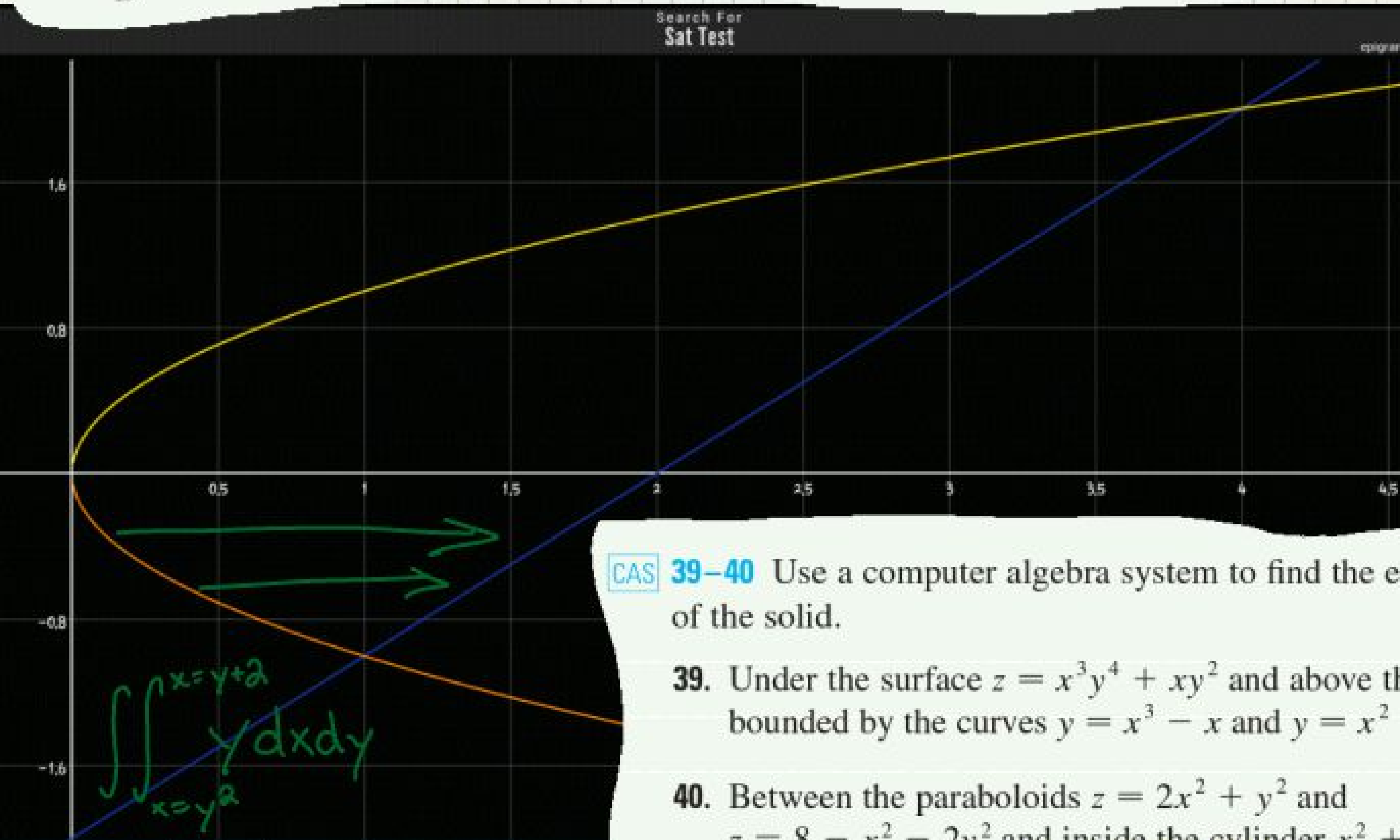
12.3

15–16 Set up iterated integrals for both orders of integration. Then evaluate the double integral using the easier order and explain why it's easier.

$y = x-2,\ y = \pm\sqrt{x}$

$x = y+2 \quad x = y^2$

15. $\iint_D y\, dA$, $\quad D$ is bounded by $y = x - 2,\ x = y^2$



$\int\int_{x=y^2}^{x=y+2} y\, dx\, dy$

CAS **39–40** Use a computer algebra system to find the exact volume of the solid.

39. Under the surface $z = x^3y^4 + xy^2$ and above the region bounded by the curves $y = x^3 - x$ and $y = x^2 + x$ for $x \geqslant 0$

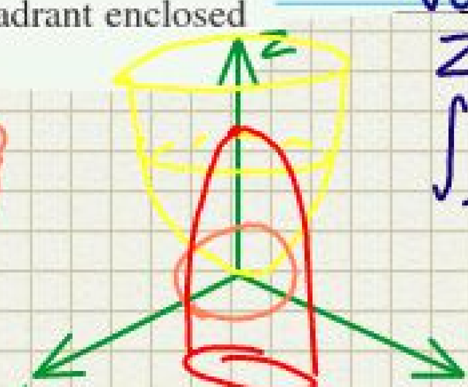
40. Between the paraboloids $z = 2x^2 + y^2$ and $z = 8 - x^2 - 2y^2$ and inside the cylinder $x^2 + y^2 = 1$

Volume beneath $z = 8-x^2-2y^2$ is

$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 8-x^2-2y^2\, dy\, dx$

- Volume below $z = 2x^2+y^2$ is

$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 2x^2+y^2\, dy\, dx$



12. $\iint_R ye^x\, dA$, where R is the region in the first quadrant enclosed by the circle $x^2 + y^2 = 25$

12.4 $= \{(r,\theta) \mid 0 \le r \le 5,\ 0 \le \theta \le 2\pi\}$

$\iint_R ye^x dA$

$= \int_0^{\frac{\pi}{2}}\int_0^5 r\sin\theta\, e^{r\cos\theta}\, r\, dr\, d\theta$

NO

$= \int_0^5\int_0^{\frac{\pi}{2}} r(r\sin\theta)e^{r\cos\theta}\, d\theta\, dr \quad = \int_0^5 r - re^r dr$

$= \int_0^5 r e^{r\cos\theta}\Big]_{\theta=0}^{\theta=\frac{\pi}{2}} dr = \int_0^5 r(e^{r\cdot 0} - e^{r\cdot 1})\, dr$

Dot & Cross

$|a \times b|$ = area of parrellogram $\vec{a} \times \vec{b}$

lines

Parameterize the things

10.5

final

dot product

cross product (area of a parellelogram = |cross product|)

equations of lines and planes

cylindrical / spherical coordinates

curvature

arc length

parametric surfaces

partial drive

max + min

lagrange mult

double integrel + convert to cylindricl coord.